Math 010 Exam 3 Study Guide

This study guide is designed to help you prepare for **Exam 3**. It categorizes the exam content into actionable skills and conceptual understanding. Given that proofs and conceptual explanations are key areas of focus, extra emphasis is placed on the underlying logic behind the procedures.

1 Vector Spaces and Axioms

What you should be able to do:

- Identify whether a set with non-standard addition or scalar multiplication satisfies the vector space axioms.
- Provide a formal demonstration to show exactly which axiom fails (e.g., distributive laws or associativity of scalar multiplication).

You should understand:

- The definition of a vector space requires all ten axioms to hold; if one fails, the set is not a vector space.
- How non-standard definitions of operations "break" specific properties.

2 The Subspace Test

What you should be able to do:

- Use the Subspace Test to determine if a subset W of R^n is a subspace.
- Check for the two necessary conditions: closure under addition and closure under scalar multiplication.

You should understand:

- Why both conditions must be checked, and a single example is not sufficient to prove closure.
- The distinction between checking closure with arbitrary vectors ($\mathbf{u}$ and $\mathbf{v}$ satisfying the constraint) versus plugging in specific numbers.

3 Linear Independence and Span

What you should be able to do:

- Determine if a specific vector belongs to the span of a set.
- Express a vector as a linear combination of other vectors if it lies within their span.
- Determine whether a set of vectors is linearly independent or dependent.

You should understand:

- Why asking “does w lie in span$\{\mathbf{v}_1, \mathbf{v}_2\}$?” is equivalent to asking whether $A\mathbf{x} = \mathbf{w}$ is consistent, with columns of A being the spanning vectors.
- Why a set is linearly dependent if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.
- The connection between linear independence and the rank of the matrix formed by the vectors.

4 Bases and Dimension

What you should be able to do:

- Prove that a set of vectors forms a basis for R^n.
- Find a basis for a subspace spanned by a given set of vectors and state its dimension.
- Identify if a set *cannot* be a basis based solely on the number of vectors relative to the dimension of the space.

You should understand:

- Why every basis for a finite-dimensional vector space must have the same number of vectors.
- The difference between a linearly independent set and a basis.

5 Change of Basis and Coordinate Vectors

What you should be able to do:

- Find the transition matrix $P_{B\to B'}$ from one basis B to another basis B'.
- Compute the **coordinate vector** $[\mathbf{v}]_B$ for a given vector $\mathbf{v}$.
- Use transition matrices to convert coordinate vectors between bases.

You should understand:

- The relationship between a vector in standard coordinates and its representation relative to a specific basis.

6 Fundamental Subspaces: Null Space, Rank, and Nullity

What you should be able to do:

- Find a basis for the null space (null(A)) of a matrix.
- Calculate the rank and nullity of a matrix.

You should understand:

- The Rank-Nullity Theorem and how it relates the dimensions of the fundamental subspaces to the number of columns in the matrix.

- How to read off free variables from the row-echelon form and write the solution set in parametric vector form to extract basis vectors for $\text{null}(A)$

7 True/False Conceptual Questions

What you should be able to do:

- Evaluate a mathematical statement about vector spaces as true or false and provide a brief but complete justification.
- For false statements, produce a specific counterexample.
- For true statements, give a short conceptual or computational argument.

8 Conceptual Proofs

Focus Area: Constructing Arguments

- Prove the uniqueness of representation for a vector relative to a basis.
- Prove properties of linear independence, such as why a set containing the zero vector is always dependent.
- Prove that adding a vector outside the span of an independent set maintains independence.

Math 010: Exam 3 Practice Worksheet

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1. Vector Space Axioms

Let $V = R^2$ with standard vector addition but scalar multiplication defined by

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \text{ for } c \geq 0, \qquad c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -cx \\ -cy \end{bmatrix} \text{ for } c < 0.$$

Determine whether the set is a vector space. If not, identify all axioms that fail.

2. Subspace Test

Use the Subspace Test to determine whether

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 : 2x - y + 3z = 0 \right\}$$

is a subspace of R^3.

3. Span and Linear Combinations

Determine whether $\mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ belongs to $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. If it does, express $\mathbf{w}$ as a linear combination.

4. Linear Independence

Determine whether $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} \right\}$ is linearly independent.

5. Bases and Dimension

a. Show that $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis for R^2.

b. Without any computation, explain why $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$ cannot be a basis for R^2.

c. Without any computation, explain why $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$ cannot be a basis for R^3.

6. True/False

Determine whether each statement is true or false. Justify briefly.

a. The span of any three nonzero vectors in R^3 is all of R^3.
b. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then one of the vectors must be the zero vector.
c. The union of two subspaces of a vector space is always a subspace.

7. Finding a Basis for a Spanned Subspace

Find a basis for the subspace of R^3 spanned by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \right\}.$$

State the dimension of the subspace.

8. Change of Basis

Let $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and let S denote the standard basis for R^2.

a. Find the transition matrix $P_{S \to B}$.
b. Use the transition matrix to find $[\mathbf{v}]_B$ for $\mathbf{v} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$.

9. Null Space, Rank, and Nullity

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Find:

a. A basis for null(A).
b. $R(A)$.
c. nullity(A). Verify the Rank-Nullity Theorem.

10. Proof Practice

Practice proving each of the following statements.

a. The intersection of two subspaces U and W of a vector space V is also a subspace of V.
b. If a set S spans a vector space V and one vector in S is a linear combination of the others, then removing that vector does not change the span.
c. If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r$ in exactly one way.
d. If a set contains the zero vector, then the set is linearly dependent.

e. The row vectors of an $n \times n$ invertible matrix A form a basis for R^n.

f. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in span$\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

g. If a vector $\mathbf{v}$ in R^n is orthogonal to each vector in a basis for a subspace W of R^n, then $\mathbf{v}$ is orthogonal to every vector in W.